Problem 10. A Uniformly Stretched Circular Plate Subjected to Axially Symmetrical Discontinuous Lateral Loads

This is identical with Problem 9 except that the plate is now subjected to a uniform membrane tension of magnitude T per unit length in all directions, instead of the compression P.

The differential equation is the same as Eq. (45) except that κ^2 is replaced by $-\tau^2$ where

$$\tau = (T/D)^{1/2} \tag{50}$$

and the solution is given by

$$y_H(r) = A_1 I_0\langle\tau r) + A_2 K_0\langle\tau r) + A_3 \tag{51}$$

$$\left.\begin{aligned} G_1(r) &= (M_0 a_1/\tau D)[K_1(\tau a_1)I_0\langle\tau r) + I_1(\tau a_1)K_0(\tau r) - (1/\tau a_1)] \\ G_2(r) &= (Wa_2/T)[K_0(\tau a_2)I_0(\tau r) - I_0(\tau a_2)K_0\langle\tau r) - \log(r/a_2)] \\ G_3(r) &= (pa_3/\tau T)[K_1(\tau a_3)I_0(\tau r) + I_1(\tau a_3)K_0(\tau r) - (1/\tau a_3) + \tfrac{1}{2}\tau a_3\{1 - (r^2/a_3{}^2) + 2\log(r/a_3)\}] \end{aligned}\right\} \tag{52}$$

where I_0, K_0 and I_1, K_1 are modified Bessel functions of the first and second kinds, of zero and unit order, respectively. The following identities have been used in deriving Eqs. (52):

$$I_0'(z) = I_1(z) \qquad K_0'(z) = -K_1(z)$$

$$\left.\begin{aligned} K_1(z)I_0(z) + I_1(z)K_0(z) &= K_1(z)I_1'(z) - I_1(z)K_1'(z) = 1/z \\ I_1'(z)K_0(z) + K_1'(z)I_0(z) &= -1/z^2 \end{aligned}\right\} \tag{53}$$

It may be noted that the heaviness of the algebra in a paper by Hicks,[9] which derives expressions for the bending moments in uniformly stretched plates due to loads similar to W and p in Fig. 2, could have been reduced considerably by the introduction of appropriate Macaulay brackets.

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Influence of Tesseral Harmonics on Nearly Circular Polar and Equatorial Orbits

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Approximate closed form expressions are obtained for the radial, cross-track, in-track, nodal period, and sidereal period perturbations due to the second sectorial ($J_2^{(2)}$) harmonic. The effects of higher-order tesseral harmonics on the position coordinates are analyzed by means of special perturbation programs. For low-altitude orbits, radial perturbations as large as 1200 ft occur due to $J_2^{(2)}$. Secular in-track perturbations (with respect to an unperturbed orbit) due to $J_2^{(2)}$ are found to be as large as 0.012°/rev, whereas cross-track perturbations are periodic with maximum amplitudes of 0.02°. Periodic oscillations occur in the orbital period with amplitudes of ±0.003 min about a mean value that differs from the unperturbed value. Of the higher-order tesserals that are investigated, the $J_3^{(3)}$ term is most influential, contributing perturbations of the same order as $J_2^{(2)}$.

Nomenclature

r = radial distance to the satellite from the center of the earth
g_r = radial component of acceleration due to the tesseral harmonics in the earth's gravity potential
g_θ = component of acceleration in the θ direction due to the tesseral harmonics
g_ϕ = component of acceleration in the ϕ direction due to the tesseral harmonics
θ = position coordinate measured in the longitudinal direction
ϕ = position coordinate measured in the direction of increasing latitude
μ = universal gravitational constant times mass of the earth
r_c = radius of reference circular orbit (const)
r_1 = perturbation in the radius due to g_r, g_θ, and g_ϕ, and the departure from circularity in the initial conditions
θ_1 = perturbation in the θ direction due to g_r, g_θ, and g_ϕ, and initial conditions
ϕ_1 = perturbation in the ϕ direction due to g_r, g_θ, and g_ϕ, and initial conditions
ϕ_c = angular position (measured from the equator in the direction of increasing latitude) that the satellite would have in the reference circular orbit; $\phi_c = \dot{\phi}_c t$
θ_c = angular position (measured in the direction of increasing longitude) that the satellite would have in the reference circular orbit; $\theta_c = \dot{\theta}_c t$
$(\dot{\ })$ = derivative with respect to time

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Introduction

THE effects of tesseral harmonics of the earth's gravity potential on the motion of artificial satellites have been studied to some extent by numerous investigators.[1-16] These studies, with the exception of those specifically dealing with the 24-hr equatorial satellite, generally describe the perturbations in the orbital elements caused by the tesseral harmonics. It is often desirable, however, to describe the satellite's position as a function of time. Since the results of the variation of the elements studies are not readily utilizable for position perturbation investigations, the analyses conducted herein are based on variation of coordinates techniques.

In order to demonstrate the more salient features of the problem under investigation and to facilitate the mathematical analyses, the study is restricted to nearly circular, polar and equatorial orbits.

Analysis

I. Equations of Motion

1. *Fundamental equations*

The equations of motion of a satellite in orbit about the earth and subjected to a gravitational potential that includes only tesseral harmonics† may be written as

$$\ddot{r} - r\dot{\theta}^2\cos^2\phi - r\dot{\phi}^2 = -(\mu/r^2) + g_r(r, \theta, \phi) \tag{1}$$

$$\frac{1}{r\cos\phi}\frac{d}{dt}(r^2\dot{\theta}\cos^2\phi) = g_\theta(r, \theta, \phi) \tag{2}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}) + r\dot{\theta}^2\cos\phi\sin\phi = g_\phi(r, \theta, \phi) \tag{3}$$

2. *Linearized equations of motion*

Since the terms g_r, g_θ, and g_ϕ are small compared to the term μ/r^2, their effects may be considered to be perturbations upon the nominal orbit obtained from central force field theory only. The nominal, or reference orbit (in the case of nearly circular orbits), may be taken as a circular orbit with perturbations in the radius and angular rate due to "initial" conditions that depart from circular orbit conditions. If the perturbations due to the tesseral harmonics are also superimposed upon the reference orbit, one may write

$$\left.\begin{aligned}
&r = r_c + r_1;\ \dot{r} = \dot{r}_1;\ r_1 \ll r_c \\
&\theta = \theta_c + \theta_1;\ \dot{\theta} = \dot{\theta}_c + \dot{\theta}_1;\ \ddot{\theta} = \ddot{\theta}_1;\ \dot{\theta}_1 \ll \dot{\theta}_c \\
&\qquad\qquad\text{(equatorial orbits)} \\
&\theta = \theta_1;\ \dot{\theta} = \dot{\theta}_1;\ \ddot{\theta} = \ddot{\theta}_1\ \text{(polar orbits)} \\
&\phi = \phi_c + \phi_1;\ \dot{\phi} = \dot{\phi}_c + \dot{\phi}_1;\ \ddot{\phi} = \\
&\qquad\qquad \ddot{\phi}_1;\ \dot{\phi}_1 \ll \dot{\phi}_c\ \text{(polar orbits)} \\
&\phi = \phi_1;\ \dot{\phi} = \dot{\phi}_1;\ \ddot{\phi} = \ddot{\phi}_1\ \text{(equatorial orbits)}
\end{aligned}\right\} \tag{4}$$

Substituting Eqs. (4) into (1–3), expanding, and neglecting second-order terms in r_1, $\dot{\theta}_1$, $\dot{\phi}_1$, and products of r_1 and $\dot{\theta}_1$, r_1 and $\dot{\phi}_1$, etc., the following is obtained:

Polar Orbits

$$\ddot{r}_1 - 3\dot{\phi}_c^2 r_1 - 2r_c\dot{\phi}_c\dot{\phi}_1 = g_{r_{\text{polar}}} \tag{5}$$

$$2\dot{\phi}_c\dot{r}_1 + (r_c + r_1)\ddot{\phi}_1 = g_{\phi_{\text{polar}}} \tag{6}$$

$$r_c^2\dot{\theta}_1\cos^2\dot{\phi}_c t = \int(r_c\cos\dot{\phi}_c t)g_{\theta_{\text{polar}}}\,dt \tag{7}$$

† Since it is desired only to assess the effects of the tesseral harmonics, zonal harmonics and other perturbations such as atmospheric drag are not included in the perturbative potential. The effect of neglecting these perturbations in the formulation is discussed in later sections.

Equatorial Orbits

$$\ddot{r}_1 - 3\dot{\theta}_c^2 r_1 = -2r_c\dot{\theta}_c\dot{\theta}_1 = g_{r_{\text{eq}}} \tag{8}$$

$$2\dot{\theta}_c\dot{r}_1 + (r_c + r_1)\ddot{\theta}_1 = g_{\theta_{\text{eq}}} \tag{9}$$

$$\ddot{\phi}_1 + \dot{\phi}_c^2\phi_1 = g_{\phi_{\text{eq}}} \tag{10}$$

where the perturbative accelerations for the polar and equatorial orbits are yet to be determined. In the development of Eqs. (5–7), use has been made of the fact that $\dot{\phi}_c^2 = \mu/r_c^3$ (for polar orbits), whereas, in obtaining Eqs. (8–10), it was recognized that $\dot{\theta}_c^2 = \mu/r_c^3$ (for equatorial orbits).

It is now desirable to uncouple Eqs. (5) and (6) and Eqs. (8) and (9). Solving Eq. (5) for $\dot{\phi}_1$, differentiating, and substituting into Eq. (6), leads to

$$\dddot{r}_1 + \dot{\phi}_c^2\dot{r}_1 = 2\dot{\phi}_c g_{\phi_{\text{polar}}} + \dot{g}_{r_{\text{polar}}} - 2\dot{\phi}_c(r_1/r_c)g_{\phi_{\text{polar}}} \tag{11}$$

$$\ddot{\phi}_1 = \frac{1}{r_c}(g_{\phi_{\text{polar}}} - 2\dot{\phi}_c\dot{r}_1) - \frac{1}{r_c}\left(\frac{r_1}{r_c}\right)(g_{\phi_{\text{polar}}} - 2\dot{\phi}_c\dot{r}_1) \tag{12}$$

The last terms on the right side of Eqs. (11) and (12) result from the term $r_1\ddot{\phi}_1$ of Eq. (6). The product of terms r_1 and $g_{\phi_{\text{polar}}}$ is of the second-order in the $J_n^{(m)}$; they are the coefficients of the harmonic terms under investigation. Since only solutions to a first-order in the $J_n^{(m)}$ are being considered, these terms are omitted from the analysis. The product of the term r_1 and $\dot{r}_1$ of Eq. (12) is also of the second-order in $J_n^{(m)}$ and shall be neglected.

Thus, Eqs. (11) and (12) become

$$\dddot{r}_1 + \dot{\phi}_c^2\dot{r}_1 = 2\dot{\phi}_c g_{\phi_{\text{polar}}} + \dot{g}_{r_{\text{polar}}} \tag{13}$$

$$\ddot{\phi}_1 = (1/r_c)(g_{\phi_{\text{polar}}} - 2\dot{\phi}_c\dot{r}_1) \tag{14}$$

A similar uncoupling of Eqs. (8) and (9) for the case of equatorial orbits leads to

$$\dddot{r}_1 + \dot{\theta}_c^2\dot{r}_1 = 2\dot{\theta}_c g_{\theta_{\text{eq}}} + \dot{g}_{r_{\text{eq}}} \tag{15}$$

$$\ddot{\theta}_1 = (1/r_c)(g_{\theta_{\text{eq}}} - 2\dot{\theta}_c\dot{r}_1) \tag{16}$$

Equations (13, 14, and 7) constitute a set of differential equations that describe the perturbations from the reference polar orbit caused by the terms in the potential whose effects are being investigated and the initial departure from circularity. Equations (15, 16, and 10) represent the perturbation equations for the case of equatorial orbits.

3. *Reduction of the linearized equations to quadratures*

Inspection of the perturbation equations obtained in the preceding section reveals that, if the perturbative functions g_r, g_θ, and g_ϕ can be specified as functions of the independent variable (time) only, the equations become linear and may be solved by standard techniques. Assuming that the perturbative accelerations can be represented explicitly in terms of the time, and applying the properties of the Laplace transform, one obtains:

Polar Orbits

$$\begin{aligned}
r_1 = {} & r_1(0)(2 - \cos\dot{\phi}_c t) + \frac{\dot{r}_1(0)}{\dot{\phi}_c}\sin\dot{\phi}_c t + \\
& 2r_c\frac{\dot{\phi}_1(0)}{\dot{\phi}_c}(1 - \cos\dot{\phi}_c t) + \frac{g_{r_{\text{polar}}}(0)}{\dot{\phi}_c^2}(1 - \cos\dot{\phi}_c t) + \\
& \frac{2}{\dot{\phi}_c}\int_0^t [1 - \cos\dot{\phi}_c(t - \tau)]g_{\phi_{\text{polar}}}(\tau)d\tau + \\
& \frac{1}{\dot{\phi}_c}\int_0^t [\sin\dot{\phi}_c(t - \tau)]g_{r_{\text{polar}}}(\tau)d\tau
\end{aligned} \tag{17}$$

$$\begin{aligned}
\phi_1 = {} & \phi_1(0) + \dot{\phi}_1(0)t + \frac{1}{r_c}\int_0^t (t - \tau)g_{\phi_{\text{polar}}}(\tau)d\tau - \\
& \frac{2\dot{\phi}_c}{r_c}\int_0^t (t - \tau)\dot{r}_1(\tau)d\tau
\end{aligned} \tag{18}$$

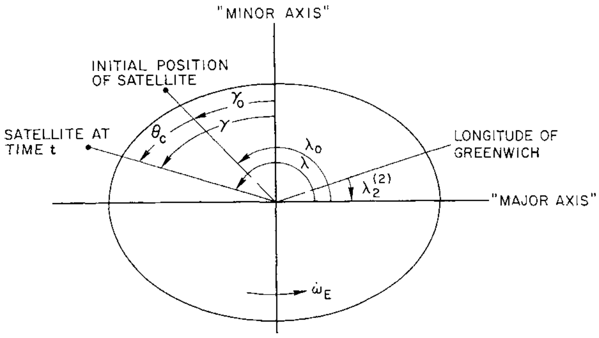

Fig. 1 Earth's equatorial section (geometry for $J_2^{(2)}$ analysis).

$$\theta_1 = \int \left\{ \frac{1}{r_c^2 \cos^2\dot{\phi}_c t} \left[\int (r_c \quad \cos\dot{\phi}_c t) g_{\theta_{\text{polar}}} dt \right] \right\} dt \quad (19)$$

Equatorial Orbits

$$r_1 = r_1(0)(2 - \cos\dot{\theta}_c t) + \frac{\dot{r}_1(0)}{\dot{\theta}_c} \sin\dot{\theta}_c t + 2r_c \frac{\dot{\theta}_1(0)}{\dot{\theta}_c} (1 - \cos\dot{\theta}_c t) + \frac{g_{r_{\text{eq}}}(0)}{\dot{\theta}_c^2} (1 - \cos\dot{\theta}_c t) + \frac{2}{\dot{\theta}_c} \int_0^t [1 - \cos\dot{\theta}_c(t - \tau)] g_{\theta_{\text{eq}}}(\tau) d\tau + \frac{1}{\dot{\theta}_c} \int_0^t [\sin\dot{\theta}_c(t - \tau)] g_{r_{\text{eq}}}(\tau) d\tau \quad (20)$$

$$\theta_1 = \theta_1(0) + \dot{\theta}_1(0)t + \frac{1}{r_c} \int_0^t (t - \tau) g_{\theta_{\text{eq}}}(\tau) d\tau - \frac{2\dot{\theta}_c}{r_c} \int_0^t (t - \tau) \dot{r}_1(\tau) d\tau \quad (21)$$

$$\phi_1 = \phi_1(0) \cos\dot{\theta}_c t + \frac{\dot{\phi}_1(0)}{\dot{\theta}_c} \sin\dot{\theta}_c t + \frac{1}{\dot{\theta}_c} \int_0^t \sin\dot{\theta}_c(t - \tau) g_{\phi_{\text{eq}}}(\tau)\, d\tau \quad (22)$$

where $r_1(0)$, $\dot{r}_1(0)$, etc., denote deviations from the reference circular orbit at the epoch of interest. These conditions may be determined at any epoch by the following procedure. Given a nearly circular orbit of radius r, velocity V, and flight path angle α, set r equal to r_c, and compute the velocity (or angular rate, $\dot{\theta}_c$ or $\dot{\phi}_c$) that would yield a circular orbit of this radius over a spherical earth; the true velocity V will have radial and circumferential components that differ from the nominal reference velocity by $\dot{\phi}_1(0)$ or $\dot{\theta}_1(0)$ and $\dot{r}_1(0)$.

II. Perturbative Function

The general form of the earth's gravity potential may be written as

$$U = \frac{\mu}{r} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_e}{r}\right)^n P_n^{(0)}(\sin\phi) + \sum_{n=2}^{\infty} \sum_{m=1}^{n} J_n^{(m)} \left(\frac{a_e}{r}\right)^n P_n^{(m)}(\sin\phi) \cos m(\lambda - \lambda_n^{(m)}) \right] \quad (23)$$

where

a_e = earth's equatorial radius
r = radial distance
λ = east longitude
ϕ = geocentric latitude
$P_n^{(m)}$ = Legendre polynomials

The first summation term of Eq. (23) represents the contribution of the zonal (latitude dependent) harmonics, whereas the last term describes the tesseral (longitude and latitude dependent) harmonics.

The values for the gravitational constants which were employed and the terms of the potential function [Eq. (23)] which were retained in this study are as follows:

a_e = 6,378,165 m (Ref. 17)
μ = 3.98603×10^{14} m³/sec² (Ref. 17)
J_2 = 1082.36×10^{-6} (Ref. 12)
J_3 = -2.566×10^{-6} (Ref. 12)
J_4 = -2.14×10^{-6} (Ref. 12)
$J_2^{(2)}$ = 4.0×10^{-6}; $\lambda_2^{(2)} = -11°$ (Ref. 13)
$J_3^{(3)}$ = 1.91×10^{-6}; $\lambda_3^{(3)} = 51°$ (Ref. 3)
$J_4^{(1)}$ = 2.64×10^{-6}; $\lambda_4^{(1)} = 164°$ (Ref. 3)
$J_4^{(2)}$ = 1.67×10^{-7}; $\lambda_4^{(2)} = 54°$ (Ref. 3)
$J_4^{(4)}$ = 5.6×10^{-8}; $\lambda_4^{(4)} = 50°$ (Ref. 3)

For a detailed analysis of the effect of the second sectorial ($J_2^{(2)}$) harmonic, it was felt that it was advantageous to obtain an analytical formulation of the problem. This procedure would afford insight into the general behavior of satellite orbits under the influence of the tesseral harmonics and could serve as a check on the results of the machine program that was utilized for other phases of this effort. The effects of some of the higher-order harmonics, such as $J_3^{(3)}$, $J_4^{(1)}$, $J_4^{(2)}$, and $J_4^{(4)}$ were studied by means of numerical integration on the computer.

The influence of the $J_3^{(1)}$ term, which may also be of importance, was not assessed because this term had not been programed on the computer at the time this work was initiated.

In order to complete the analytical formulation of the $J_2^{(2)}$ effect on the polar and equatorial orbits, it is now necessary to develop the perturbing functions in a form suitable for substitution into Eqs. (17–22). From Eq. (23), retaining only the $J_2^{(2)}$ term, and recalling that

$$g_r = \frac{\partial U}{\partial r} \qquad g_\theta = \frac{1}{r \cos\phi} \frac{\partial U}{\partial \theta} \qquad g_\phi = \frac{1}{r} \frac{\partial U}{\partial \phi}$$

the following is obtained:

$$g_r = \frac{9J_2^{(2)}\mu a_e^2}{r^4} \cos^2\phi \cos 2\gamma \quad (24)$$

$$g_\theta = \frac{6J_2^{(2)}\mu a_e^2}{r^4} \cos\phi \sin 2\gamma \quad (25)$$

$$g_\phi = \frac{6J_2^{(2)}\mu a_e^2}{r^4} \cos\phi \sin\phi \cos 2\gamma \quad (26)$$

where γ is the east longitude from earth's "minor axis" (Fig. 1).

a) Polar orbits: From Fig. 1, it may be seen that, for polar orbits, the longitude of the satellite γ at any time is given by

$$\gamma = \gamma_0 - \dot{\omega}_E t$$

where $\dot{\omega}_E$ is the rate of the earth's angular rotation. The latitude ϕ of the satellite is

$$\phi = \phi_c + \phi_1 = \dot{\phi}_c t + \phi_1$$

Substituting these values of γ and ϕ into Eqs. (24–26), the perturbative accelerations become

$$g_{r_{\text{polar}}} = \frac{9J_2^{(2)}\mu a_e^2}{r_c^4} \cos^2\dot{\phi}_c t \cos 2(\gamma_0 - \dot{\omega}_E t) \quad (27)$$

$$g_{\theta_{\text{polar}}} = \frac{6J_2^{(2)}\mu a_e^2}{r_c^4} \cos\dot{\phi}_c t \sin 2(\gamma_0 - \dot{\omega}_E t) \quad (28)$$

$$g_{\phi_{\text{polar}}} = \frac{3J_2^{(2)}\mu a_e^2}{r_c^4} \sin 2\dot{\phi}_c t \cos 2(\gamma_0 - \dot{\omega}_E t) \quad (29)$$

where simplifications of the following nature were effected:

$$\cos\phi = \cos(\dot{\phi}_c t + \phi_1) = \cos\dot{\phi}_c t \cos\phi_1 - \sin\dot{\phi}_c t \sin\phi_1$$

$$\cos\phi \cong \cos\dot{\phi}_c t - \phi_1 \sin\dot{\phi}_c t \text{ (where } \phi_1 \text{ is small)}$$

$$J_2^{(2)} \cos\phi = J_2^{(2)} \cos\dot{\phi}_c t$$

Also products of r_1 and $J_2^{(2)}$ were neglected.

b) Equatorial orbits: In the case of equatorial orbits, the perturbative acceleration equations become

$$g_{r_{\mathrm{eq}}} = \frac{9J_2^{(2)}\mu a_e^2}{r_c^4} \cos 2[(\dot{\theta}_c - \dot{\omega}_E)t + \gamma_0] \tag{30}$$

$$g_{\theta_{\mathrm{eq}}} = \frac{6J_2^{(2)}\mu a_e^2}{r_c^4} \sin 2[(\dot{\theta}_c + \dot{\omega}_E)t + \gamma_0] \tag{31}$$

$$g_{\phi_{\mathrm{eq}}} = 0 \tag{32}$$

where it was recognized that

$$\gamma = \theta_c + \gamma_0 - \dot{\omega}_E t = \dot{\theta}_c t + \gamma_0 - \dot{\omega}_E t$$

$$\phi = \phi_1 \qquad \cos\phi_1 \cong 1 \qquad \sin\phi_1 \cong \phi_1$$

It is now possible to substitute Eqs. (27–32) into Eqs. (17–22) to obtain the perturbation in the motion of the satellite due to the effects of $J_2^{(2)}$.

III. Perturbed Motion of the Satellite due to $J_2^{(2)}$

1. *Polar orbits*

Substitution of Eqs. (27–29) into (17–19) yields the perturbations in the radius, in-track, and cross-track positions of the satellite due to the effect of $J_2^{(2)}$. The integration of Eqs. (17) and (18) is straightforward although extremely tedious; the final results are lengthy and are not given here. (See Ref. 18 for the complete results.) The integration of Eq. (19) presents some conceptual difficulties, and further discussion is reserved for later.

a) Radial perturbation: The results of the integrations[18] indicate that the radial perturbation due to the $J_2^{(2)}$ term consists of short- and long-period oscillations. The short-period oscillations have periods equal to one revolution of the satellite, whereas the long-period oscillations have periods that are combinations of the satellite's period and the earth's period. A strong dependency of the radial variation on the longitude of injection is also evident.

The dependence of the radial perturbations on the *latitude* of injection can not be determined from the integrated results since the development assumed injection over the equator. Preliminary investigations, however, appear to indicate that the behavior of the radial perturbations is also a strong function of the injection latitude. Figure 2 shows the short- and long-period variation for some representative polar orbits. The envelopes in Fig. 2 were plotted by drawing a curve through the peak values that occurred every half revolution.

b) In-track perturbation: The complete solution for the perturbation in the in-track position due to the $J_2^{(2)}$ term is given in Ref. 18.

It was found that the in-track perturbation due to $J_2^{(2)}$ consists of secular, short-period, and long-period terms. The secular terms arise since the period of a satellite moving over a force model containing $J_2^{(2)}$ differs from that of a satellite that is not subjected to the perturbations of $J_2^{(2)}$. The short-period terms are of the satellite's period, whereas the long-period terms are of periods that are combinations of the satellite's period and the earth's rotational period.

The secular in-track perturbation contributed by $J_2^{(2)}$ is shown in Fig. 3 as a function of orbit altitude and injection longitude. As indicated in the figure, these results were obtained for a polar, initially circular orbit launched over the equator. It is of interest to note in Fig. 3 that, for injection at $\gamma_0 = 45°$ (or $\gamma_0 = 135°$), the secular in-track perturbation vanishes and only periodic oscillations remain.

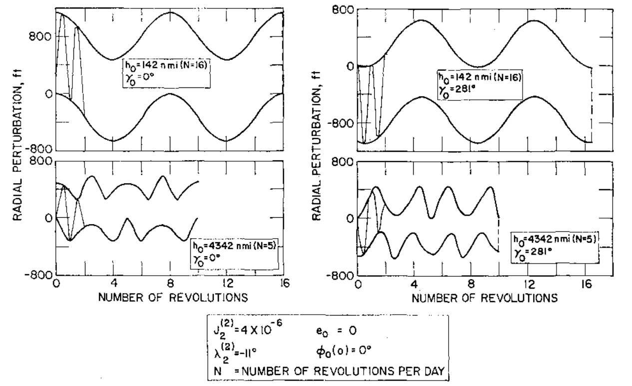

Fig. 2 Short- and long-period radial perturbation contributed by the $J_2^{(2)}$ term—polar orbits.

It should be remarked that the results of Fig. 3 were obtained for injection at zero latitude. The analytical results were developed under this assumption and can not be utilized to obtain the latitude dependency of the in-track variation.

It is clear from inspection of Fig. 3 that the initial longitude of the satellite strongly influences the subsequent behavior of the satellite. Furthermore, it is evident from the equations of Ref. 18 that, for any given longitude of injection, the initial conditions (radius and/or velocity) can be chosen so as to eliminate the secular terms in the in-track perturbation. That is, initial conditions may be determined for which the mean value of the period of the unperturbed orbit is the same as the period of the perturbed orbit.

The final equations contain indeterminacies for several values of $\dot{\phi}_c/\dot{\omega}_E$. Most notable is $\dot{\phi}_c/\dot{\omega}_E = 1$, which is the 24-hr satellite. This special case is discussed in detail in Ref. 18.

c) Cross-track perturbation: Substitution of the perturbative cross-track acceleration [Eq. (28)] into Eq. (19) yields, in principle, the cross-track perturbations due to the $J_2^{(2)}$ term. The indicated substitution, however, leads to a formidable integral that does not appear to be solvable in terms of elementary functions. A considerable simplifi-

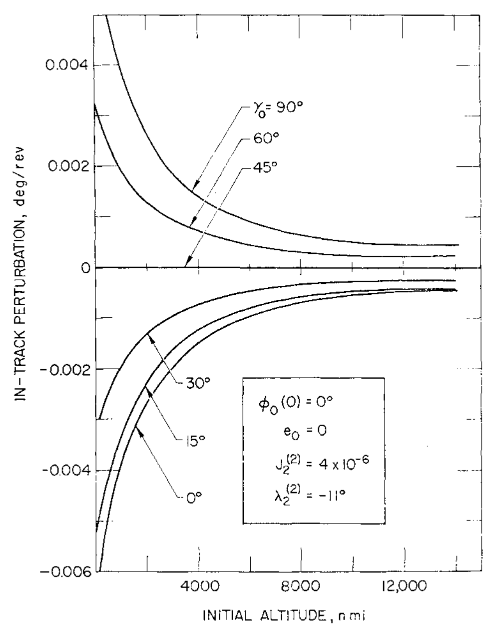

Fig. 3 Secular in-track perturbation due to $J_2^{(2)}$—polar orbits.

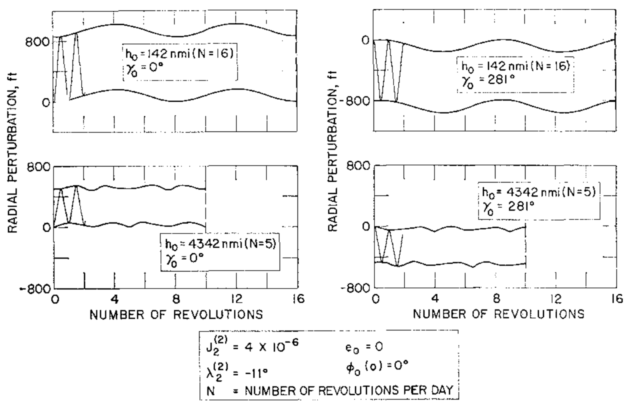

Fig. 4 Short- and long-period radial perturbation contributed by the $J_2^{(2)}$ term—equatorial orbits.

cation can be effected if it is assumed that the longitude of the satellite trace does not change significantly during one revolution. In effect, one substitutes the average value of the longitudinal position described by the satellite during the time period of interest for the actual time-dependent longitudinal variation. This assumption is reasonably correct for low-altitude satellites and leads to

$$\theta_1 = \int \left\{ \frac{1}{r_c \cos^2 \dot{\phi}_c t} \left[\frac{6J_2^{(2)} \mu a_e^2}{r_c^3} \sin 2\gamma_{av} \int (\cos^2 \dot{\phi}_c t) dt \right] \right\} dt \tag{33}$$

Solution of Eq. (33) yields

$$\theta_1 = 3J_2^{(2)}(a_e/r_c)^2[\phi_c \tan\phi_c + \tfrac{3}{4} \ln(\cos\phi_c)] \sin 2\gamma_{av} \tag{34}$$

Equation (34) may be utilized to determine the cross-track perturbations due to $J_2^{(2)}$ during any one revolution of the satellite. Inspection of the equation indicates that the cross-track perturbation is zero at equatorial crossings and undetermined at latitudes of $\pm 90°$ ("longitude" or "cross-track" is undefined at the poles) for any value of γ_{av}. At high latitudes (around 87°) the magnitude of the cross-track perturbation is of the order of 0.01° to 0.02°, depending on the value of γ_{av}.

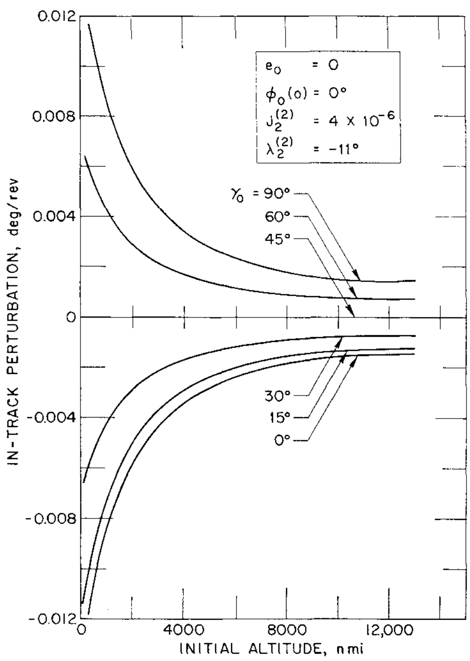

Fig. 5 Secular in-track perturbation due to $J_2^{(2)}$—equatorial orbits.

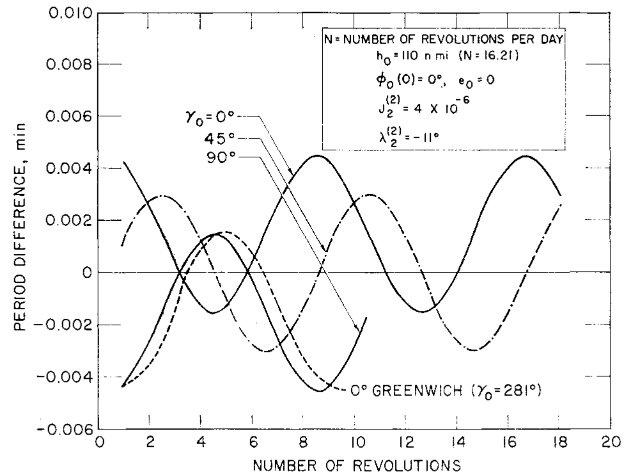

Fig. 6 Contribution to the nodal period by the $J_2^{(2)}$ term —polar orbit.

2. *Equatorial orbits*

The perturbations in the radius, in-track, and cross-track position due to $J_2^{(2)}$ for equatorial orbits are obtained by substitution of Eqs. (30–32) into Eqs. (20–22).

a) Radial perturbation: The complete solution for the perturbation in the radial position of the satellite is given in Ref. 18.

The radial perturbation was found to consist of short- and long-period variations. The short- and long-period perturbations in the radius are shown in Fig. 4. Again, it is of interest to note the strong dependency on the initial longitude.

Comparison of Figs. 3 and 4 indicates that the magnitudes of the radial perturbations are smaller for the equatorial orbits than for the polar orbits at equivalent injection longitudes.

b) In-track perturbation: The perturbation in the in-track position of the satellite due to the presence of the $J_2^{(2)}$ term is also given in Ref. 18.

As in the case of polar orbits, the in-track perturbations to the nominal equatorial orbit contain secular, short-period, and long-period terms. The secular in-track perturbation is shown as a function of altitude in Fig. 5 for various injection longitudes. It should be noted that these secular in-track perturbations are approximately twice as large as for polar orbits (see Fig. 3). The secular perturbations are removed when $\gamma_0 = 45°$ (as with polar orbits), or by the proper choice of injection radius or velocity as determined from the equations of Ref. 18.

c) Cross-track perturbation: The cross-track perturbation due to the $J_2^{(2)}$ term is obtained by substituting Eq. (32) into Eq. (22), which yields

$$\phi_1 = \phi_1(0) \cos\dot{\theta}_c t + \frac{\dot{\phi}_1(0)}{\dot{\theta}_c} \sin\dot{\theta}_c t \tag{35}$$

Since the perturbative acceleration out of plane is zero [Eq. (32)], perturbations out of the equatorial plane occur only when initial disturbances out of the plane of the orbit are introduced.

IV. Nodal and Sidereal Period Variations due to $J_2^{(2)}$

Another parameter of interest in understanding the general behavior of satellites perturbed by the $J_2^{(2)}$ term, and one which is of great consequence in tracking studies, is the period of the satellite. The "types" of periods which are felt to be most meaningful for this study are the nodal period for polar orbits and the sidereal period for the equatorial orbits. The nodal period is taken to be the time elapsed from one

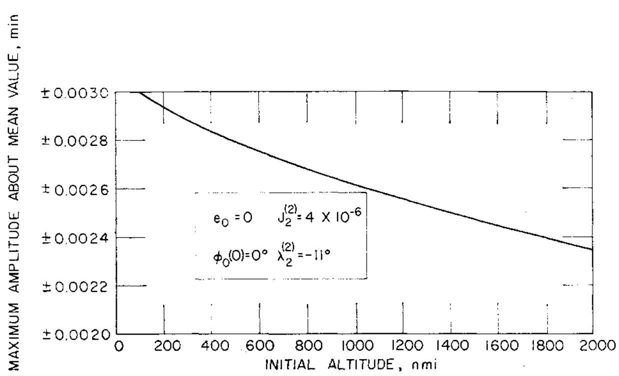

Fig. 7 Maximum amplitude of nodal period variation contributed by the $J_2^{(2)}$ term—polar orbits.

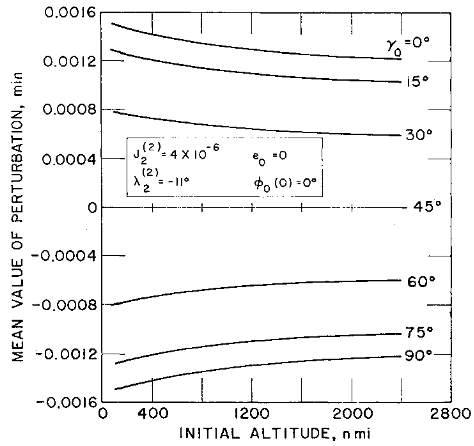

Fig. 8 Mean value of nodal period variation contributed by the $J_2^{(2)}$ term—polar orbits.

northward crossing of the equator to the next successive northward crossing. Sidereal period is the time required for the satellite to move from a point fixed in a nonrotating frame through 2π rad to the same point.

The perturbation in the period due to the $J_2^{(2)}$ term may be developed from the perturbed angular rate. Thus, for polar orbits, the time required to move through the nth revolution since epoch is given by

$$T_N = \int_{2\pi(n-1)}^{2n\pi} \frac{dt}{d\phi}\, d\phi \qquad n = 1, 2, 3, \ldots \tag{36}$$

where T_N = nodal period of polar orbit. For equatorial orbits,

$$T_S = \int_{2\pi(n-1)}^{2n\pi} \frac{dt}{d\theta}\, d\theta \qquad n = 1, 2, 3, \ldots. \tag{37}$$

where T_S is the sidereal period of equatorial orbit.

1. *Polar orbits*

Recalling that $\dot{\phi} = \dot{\phi}_1 + \dot{\phi}_c$, and substituting into Eq. (36) (where $\dot{\phi}_1$ is obtained from Ref. 18), leads to the result for the nodal period. Figure 6 shows the variation of the nodal period for a typical initially circular low-altitude orbit. It is of interest to note that the nodal period curves have a periodicity of half a day with amplitudes about the mean value which are independent of the longitude of injection. The mean value (over one cycle) is strongly dependent on injection longitude. For γ_0 of 45°, the mean value is seen to be zero, which implies that there is no secular in-track perturbation due to $J_2^{(2)}$ for this longitude; this result has been discussed previously.

Figure 7 shows the dependency of the maximum amplitude about the mean value on the initial altitude. The mean value over one cycle of the nodal period curve (one-half day) is shown in Fig. 8 as a function of altitude.

The results of Figs. 6–8 have been compared with data from numerical integration of the equations of motion. The results of the analytical formulation were found to be within 1% of the numerical results after two days of orbiting. Furthermore, numerical studies were conducted in which the model containing the $J_2^{(2)}$ term also included the J_2, J_3, and J_4 zonal harmonics, whereas the reference ("unperturbed") model contained only the J_2, J_3, and J_4 zonals. Essentially the same results as in Fig. 6 were obtained which indicates that the perturbative effects are linearly superposable, and coupling is negligible over the time period of interest here.

2. *Equatorial orbits*

Performing the operation indicated by Eq. (37) leads to the solution for the sidereal period.[18]

The perturbation to the sidereal period due to $J_2^{(2)}$ is shown in Fig. 9 for a typical example. These results compare to within 1% after two days of orbiting with numerical integration results.

It is of interest to note that the amplitudes about the mean values are considerably smaller than for polar orbits (Fig. 6), although the mean value of the perturbation is greater. It is for the latter reason that the in-track perturbation is greater for equatorial orbits than for polar orbits. The mean value of the perturbation in the sidereal period is shown in Fig. 10 for other altitudes. As in the case of polar orbits, the mean value of the perturbation is zero when γ_0 is equal to 45°.

V. Effects of Higher-Order Tesseral Harmonics

1. *In-track perturbation on polar orbits*

In view of the relatively large effect of the $J_2^{(2)}$ term on satellite orbits, a study has been conducted to determine the effects of the inclusion or omission of some of the higher-order tesseral harmonics in the earth's gravity potential on low altitude, polar satellite orbits. The investigation was conducted by means of the numerical integration of the appropriate equations of motion. The numerical integration was carried out using an Encke formulation with true anomaly as the independent variable and a Gauss-Jackson integration subroutine. The effects of any particular combination of higher-order tesseral terms were assessed by comparing the in-track position (latitude) of the satellite with the in-track position of a reference satellite at equal times.

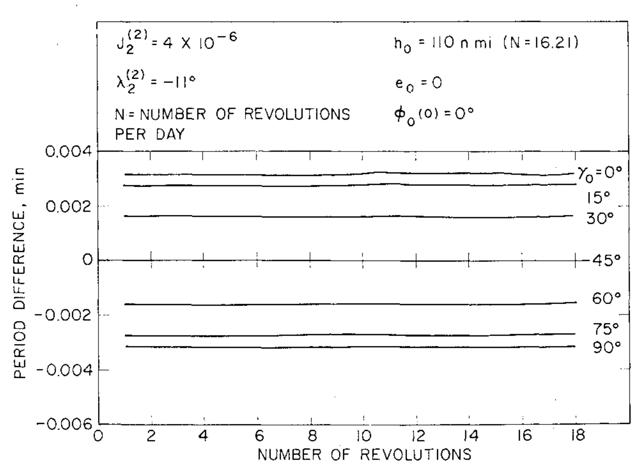

Fig. 9 Contribution to the sidereal period by the $J_2^{(2)}$ term—equatorial orbit.

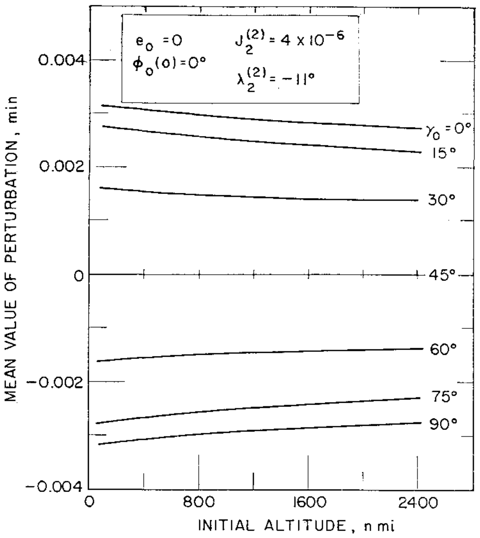

Fig. 10 Mean value of sidereal period variation contributed by the $J_2^{(2)}$ term—equatorial orbits.

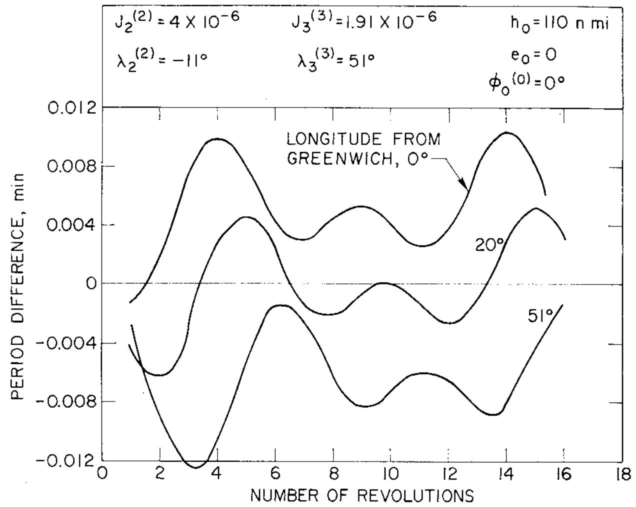

Fig. 11 Contribution to the nodal period by the $J_2^{(2)}$ and $J_3^{(3)}$ terms in combination—polar orbit.

The gravitational model that was used for reference purposes was one that included those terms whose coefficients are designated by J_2, J_3, J_4, and the primary tesseral harmonic $J_2^{(2)}$. The higher-order tesseral harmonics whose effects are reported herein are those having the coefficients $J_3^{(3)}$, $J_4^{(1)}$, $J_4^{(2)}$, and $J_4^{(4)}$. The form of the potential function which was used is given by Eq. (23); the constants that were utilized were listed earlier. In addition to the gravitational model just described, the ARDC 1959 Standard Atmosphere was utilized throughout the study and a W/C_DA of 50 was assumed.

To determine the effect of the inclusion or omission of the higher-order tesseral harmonics, a complete set of combinations of zeros and nonzeros for the $J_3^{(3)}$, $J_4^{(1)}$, $J_4^{(2)}$, and $J_4^{(4)}$ harmonics was examined. This set of combinations is shown in Table 1. The $J_3^{(1)}$ term, which may be significant, had not been incorporated into the machine program at the time of this study and was not analyzed.

In order to determine the extent to which coupling with the atmosphere affected the results, the effect of the $J_3^{(3)}$ term was also analyzed with the atmosphere deleted. Table 1 illustrates the results of including or omitting any of the higher-order tesseral harmonics for a specific orbit. A positive sign indicates that the satellite under investigation would lead the reference satellite at equal times.

Inspection of the table reveals that the $J_3^{(3)}$ term has a significant effect on the motion of a satellite; it contributes an in-track displacement of about 1° or 62 naut mile after 2 days of orbiting. The next most significant effect is that of $J_4^{(4)}$ while the contributions of $J_4^{(1)}$ and $J_4^{(2)}$ are relatively small.

The effect of $J_3^{(3)}$ has also been assessed without the inclusion of the atmosphere, and its effect is shown in Table 1 (case 5). It is noted that the atmosphere increases the in-track perturbation due to the $J_3^{(3)}$ term. After 2 days of orbiting, the in-track position over a gravitational model containing the atmosphere is 14% greater than over a model that does not account for atmospheric effects. These differences will, of course, diminish with increasing orbit altitude.

It may also be gleaned from an analysis of the results (Table 1) that the effect of each tesseral harmonic is linearly superposable even when the atmosphere is included. For example, it is seen from case 5 that $J_3^{(3)}$ alone contributes

Table 1 In-track displacement difference due to higher-order tesseral harmonics; difference measured relative to reference orbit (case 16)[a]

	Term of potential					In-track displacement difference, deg; Time from injection, min				
Case no.	$J_2^{(2)}$	$J_3^{(3)}$	$J_4^{(1)}$	$J_4^{(2)}$	$J_4^{(4)}$	700	1400	2100	2800	3500
1	X	X	X	X	X	−0.224	−0.478	−0.739	−1.015	−1.344
2	X	X	X	X	0	−0.197	−0.425	−0.657	−0.903	−1.199
3	X	X	X	0	0	−0.202	−0.433	−0.670	−0.919	−1.220
4	X	X	X	0	X	−0.228	−0.486	−0.752	−1.034	−1.370
5	X[b]	X	0	0	0	−0.196	−0.408	−0.615	−0.812	−1.042
	X	X	0	0	0	−0.202	−0.433	−0.671	−0.922	−1.225
6	X	X	0	0	X	−0.228	−0.486	−0.752	−1.032	−1.368
7	X	X	0	X	X	−0.224	−0.478	−0.740	−1.017	−1.347
8	X	X	0	X	0	−0.197	−0.425	−0.657	−0.903	−1.200
9	X	0	X	X	X	−0.022	−0.045	−0.069	−0.095	−0.125
10	X	0	X	X	0	+0.0043	+0.0088	+0.0134	+0.0186	+0.0239
11	X	0	X	0	0	+0.0002	+0.0003	+0.0002	+0.0001	−0.0004
12	X	0	X	0	X	−0.0263	−0.0533	−0.0819	−0.1121	−0.1459
13	X	0	0	X	X	−0.0223	−0.0452	−0.0692	−0.0949	−0.1235
14	X	0	0	X	0	+0.0041	+0.0085	+0.0131	+0.0181	+0.0232
15	X	0	0	0	X	−0.0264	−0.0536	−0.0822	−0.1132	−0.1483
16	X	0	0	0	0	0	0	0	0	0

[a] 110-naut mile alt; injection longitude = 0°.
[b] These results were obtained with the atmosphere deleted.

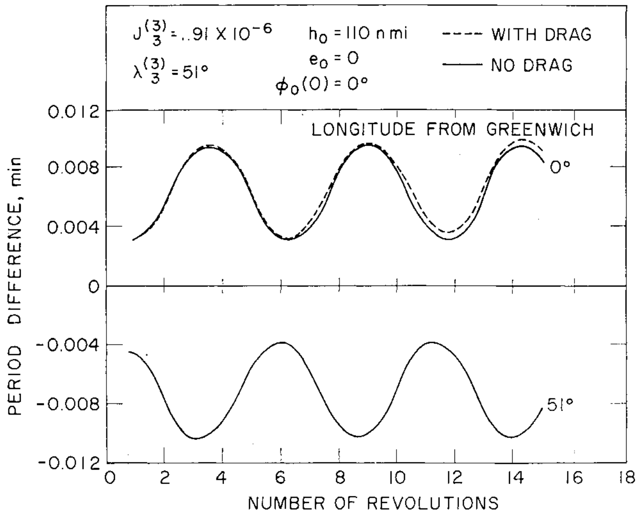

Fig. 12 Contribution to the nodal period by the $J_3^{(3)}$ term—polar orbit.

an in-track displacement of 0.202° after 700 min; thus, adding 0.202° to the result of case 9, for example, should yield the same result as in case 1, which differs from case 9 only by the inclusion of $J_3^{(3)}$. It is noted that this is indeed so. A check for all the other cases reveals that linear superposition holds rather well throughout the time period recorded in the table.

2. *Nodal period variations due to $J_3^{(3)}$*

Since it was determined that the $J_3^{(3)}$ term had the largest effect on in-track position of all the tesserals studied, it was felt that the determination of the nodal period variation due to this term would be of interest.

The nodal period variation contributed by $J_2^{(2)}$ and $J_3^{(3)}$ in combination was determined by numerical integration using a program that included J_2, J_3, and J_4 but no atmosphere. That is, a model containing the zonal harmonics was used as a reference while a model with the zonals and the two tesserals was used as the test case. Similarly the effect of $J_3^{(3)}$ alone was determined by deleting the $J_2^{(2)}$ term.

Figure 11 shows the contribution to the nodal period by the $J_2^{(2)}$ and $J_3^{(3)}$ terms combined for different longitudes (measured from Greenwich) of injection. Figure 12 shows the nodal period perturbation due to the $J_3^{(3)}$ term only. The differences in the period perturbations between the results of Figs. 11 and 12 yield the contribution of the $J_2^{(2)}$ term only. Results thus obtained for the nodal period perturbation due to $J_2^{(2)}$ have been compared with data in which the $J_2^{(2)}$ effect was determined directly, i.e., the differences in nodal periods between a model containing J_2, J_3, J_4, and $J_2^{(2)}$ and a model of J_2, J_3, and J_4 only were determined.

These comparisons yielded very nearly equal results, which indicates the effects of $J_2^{(2)}$ and $J_3^{(3)}$ are also linearly superposable. The perturbations in the nodal period due to the $J_2^{(2)}$ term obtained here have been compared with the analytical and numerical results in which no zonal harmonics had been included in the gravitational model. Again, the results were very nearly equal which implies that the zonal harmonic effects are also linearly superposable over the time periods of interest here.

Figure 12, incidentally, also shows the variation in the nodal period due to $J_3^{(3)}$ when atmospheric drag is included. The differences in the nodal period variations between the drag and no-drag case account for the differences in the in-track positions shown in Table 1 (case 5).

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